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ABSTRACT PROPERTIES P SUCH THAT ANY SEMIGROUP WHICH IS A SEMILATTICE OF COMMUTATIVE SEMIGROUPS WITH P IS COMMUTATIVE

By Miyuki Yamada

Let $P_1(G)$ and $P_2(G)$ be abstract properties pertaining to commutative semigroups G in the sense of P. M. Cohn[3]. $P_1(G)$ is said to be weaker than or equal to $P_2(G)$ and denoted by $P_1(G) \leq P_2(G)$ if and only if, for any commutative semigroup S , $P_1(G)$ is satisfied by S (i.e., $P_1(S)$ is true) whenever $P_2(G)$ is satisfied by S . If $P_1(G) \leq P_2(G)$ and $P_2(G) \leq P_1(G)$, then $P_1(G)$ and $P_2(G)$ are said to be equivalent and denoted by $P_1(G) \equiv P_2(G)$. If $P_1(G) \equiv P_2(G)$, we regard $P_1(G)$ and $P_2(G)$ as the same property. When S is a semigroup which is a semilattice of commutative semigroups S_ξ , $\xi \in \mathcal{X}$, S is not necessarily commutative. However, there is an abstract property $P(G)$ pertaining to commutative semigroups G such that any semigroup which is a semilattice of commutative semigroups with $P(G)$ is commutative. Such an abstract property is called a fully c-invariant property (abbrev., f.c.i.-property). For example, it is well-known that the property $P(G)$ " G is a group " is an f.c.i.-property. There is no greatest (i.e., weakest) f.c.i.-property with respect to the ordering relation defined above, but there is a maximal f.c.i.-property. Further, a maximal f.c.i.-property is not unique. The main purpose of this paper is to obtain maximal f.c.i.-properties, and some relevant results. All results are given without proofs.

§ 1. Introduction. A commutative idempotent semigroup Γ is called a semilattice. Define an ordering relation on Γ as follows :

(1. 1) $\alpha \leq \beta$ if and only if $\alpha\beta = \beta\alpha = \beta$.

Then, it is obvious that Γ is a partially ordered set with respect to \leq . If $\alpha \leq \beta$ and $\alpha \neq \beta$, then we shall denote it by $\alpha < \beta$. If Γ

is a totally ordered set with respect to \leq , then Γ is called a chain. Now, let $\{S_\gamma : \gamma \in \Gamma\}$ (Γ : a semilattice) be a collection of semigroups S_γ . Then, each S_γ is called the γ -component of this collection. If γ is not a minimal element of Γ (i.e., if there is an element $\alpha \in \Gamma$ such that $\alpha < \gamma$), then the corresponding S_γ is called a multiple-component. Let $S = \sum \{S_\gamma : \gamma \in \Gamma\}$ (hereafter, \sum and $+$ denote disjoint sum). If \circ is multiplication in S such that

- (1. 2) $S(\circ)$ is a semigroup, and each S_γ ($\gamma \in \Gamma$) is embedded in $S(\circ)$, i.e., $x \circ y = xy$ for all $x, y \in S_\gamma$,

and

- (1. 3) $S_\alpha \circ S_\beta \subset S_{\alpha\beta}$ for all $\alpha, \beta \in \Gamma$,

then the resulting system $S(\circ)$ is called a composition of $\{S_\gamma : \gamma \in \Gamma\}$ (with respect to Γ). Further, next we shall generalize this concept as follows: Let $\{S_\xi : \xi \in \chi\}$ (χ : a set) be a collection of semigroups S_ξ . Define multiplication $*$ in χ and multiplication \circ in $S = \sum \{S_\xi : \xi \in \chi\}$ such that $\chi(*)$ is a semilattice [chain] and $S(\circ)$ is a composition of $\{S_\xi : \xi \in \chi(*)\}$. In this case, $S(\circ)$ is called a semilattice [linear] composition of $\{S_\xi : \xi \in \chi\}$. Let $\{S_\gamma : \gamma \in \Gamma\}$ (Γ : a semilattice) be a collection of commutative semigroups S_γ . Then, sometimes there exists a composition $S(\circ)$ of $\{S_\gamma : \gamma \in \Gamma\}$ which is commutative. In this case, we shall call $S(\circ)$ a commutative composition of $\{S_\gamma : \gamma \in \Gamma\}$. Similarly if a semilattice [linear] composition $S(\circ)$ of a collection $\{S_\xi : \xi \in \chi\}$ (χ : a set) of commutative semigroups S_ξ is commutative, then $S(\circ)$ is called a commutative semilattice [linear] composition of $\{S_\xi : \xi \in \chi\}$. In general, for a given collection $\{S_\gamma : \gamma \in \Gamma\}$ (Γ : a semilattice) of semigroups S_γ , there is not necessarily a composition of $\{S_\gamma : \gamma \in \Gamma\}$ (see Yamada [4]). If there exists at least one composition of $\{S_\gamma : \gamma \in \Gamma\}$, then the collection $\{S_\gamma : \gamma \in \Gamma\}$ is said to be composable. If Γ is a chain,

then it is well-known that $\{S_\gamma : \gamma \in \Gamma\}$ is necessarily composable (e.g., see Clifford [1]). For any given collection $\{S_\gamma : \gamma \in \Gamma\}$ (Γ : a semilattice) of commutative semigroups S_γ , a composition of $\{S_\gamma : \gamma \in \Gamma\}$ is (even if it exists) not necessarily commutative. This can be seen from the following simple example :

Let $\Gamma = \{\alpha, \beta\}$ ($\alpha\beta = \beta\alpha = \beta, \alpha \neq \beta$) be a chain, S_α a commutative semigroup, and S_β a null semigroup containing at least two elements. Let $S = S_\alpha + S_\beta$, and define multiplication \circ in S as follows :

$$x \circ y = \begin{cases} xy & \text{if } x, y \in S_\alpha \text{ or } \in S_\beta, \\ y & \text{if } x \in S_\alpha, y \in S_\beta, \\ 0 & \text{if } x \in S_\beta, y \in S_\alpha, \end{cases}$$

where 0 is the zero element of S_β . Then $S(\circ)$ is a non-commutative composition of $\{S_\alpha, S_\beta\}$ with respect to Γ . In §2, we shall give a necessary and sufficient condition for a collection $\{S_\gamma : \gamma \in \Gamma\}$ (Γ : a semilattice) of commutative semigroups S_γ to be composable. Further, in the case where $\{S_\gamma : \gamma \in \Gamma\}$ is composable, we shall give a method of construction of all compositions of $\{S_\gamma : \gamma \in \Gamma\}$. We also give a necessary and sufficient condition for $\{S_\gamma : \gamma \in \Gamma\}$ that every composition of $\{S_\gamma : \gamma \in \Gamma\}$ (if it exists) be necessarily commutative. Let $P(G)$ be a proposition pertaining to commutative semigroups G . As in Cohn[3], $P(G)$ is said to be an abstract property (pertaining to commutative semigroups) if and only if $P(G)$ is invariant under isomorphism, i.e.

- (1. 4) for any commutative semigroups S_1, S_2 such that $S_1 \cong S_2$ (S_1 is isomorphic with S_2), $P(S_1)$ is true whenever $P(S_2)$ is true and vice-versa.

If $P(S)$ is true for a commutative semigroup S , then we shall say that S satisfies $P(G)$. In this case, we also say that S is a commutative semigroup with $P(G)$. For example, the properties " G is a group " and " G is cancellative " pertaining to commutative semigroups G are

abstract properties. Let $P_1(G)$ and $P_2(G)$ be abstract properties. Then $P_1(G)$ and $P_2(G)$ are said to be equivalent (denoted by $P_1(G) \equiv P_2(G)$) if the following is fulfilled:

- (1. 5) For any commutative semigroup S , $P_1(S)$ is true if and only if $P_2(S)$ is true.

Hereafter, we shall consider $P_1(G)$, $P_2(G)$ as the same property if they are equivalent. Define an ordering relation on the set \mathcal{P} of abstract properties as follows: Let $P_1(G)$ and $P_2(G)$ be abstract properties. $P_1(G) \leq P_2(G)$ if the following (1. 6) is fulfilled:

- (1. 6) For every commutative semigroup S , $P_2(S)$ is true whenever $P_1(S)$ is true.

If $P_1(G) \leq P_2(G)$ and $P_1(G) \not\equiv P_2(G)$, then the property $P_2(G)$ is said to be weaker than the property $P_1(G)$ and denoted by $P_1(G) < P_2(G)$.

It is obvious that \mathcal{P} is a partially ordered set with respect to this relation \leq (when we regard properties $P_1(G)$ and $P_2(G)$ as the same property if $P_1(G) \equiv P_2(G)$).

Next, consider the following propositions concerning an abstract property $P(G)$:

- (1. 7) For any collection $\{S_r : r \in \Gamma\}$ (Γ : a chain) of commutative semigroups S_r , where each multiple-component S_α satisfies $P(G)$, every composition of $\{S_r : r \in \Gamma\}$ is commutative.

- (1. 8) For any collection $\{S_r : r \in \Gamma\}$ (Γ : a semilattice) of commutative semigroups S_r , where each multiple-component S_α satisfies $P(G)$, every composition of $\{S_r : r \in \Gamma\}$ (if it exists) is commutative.

If (1. 7) or (1. 8) is true for $P(G)$, then $P(G)$ is called a linearly c-extensible property (abbrev., l.c.e.-property) or a fully c-extensible property (abbrev., f.c.e.-property) respectively. For example, the abstract property " G is a group " pertaining to commutative semigroups G is an f.c.e.-property. By the definitions an

f.c.e.-property is clearly an l.c.e.-property, but the converse is not true (see Remark below). In § 3, we shall show existence of the weakest l.c.e.-property and the weakest f.c.e.-property and try to determine these properties.

Next, consider also the following propositions concerning an abstract property $P(G)$:

(1. 9) For any collection $\{S_\xi ; \xi \in \lambda\}$ (λ : a set) of commutative semigroups S_ξ , where each S_ξ satisfies $P(G)$, every linear composition of $\{S_\xi : \xi \in \lambda\}$ is commutative.

(1. 10) For any collection $\{S_\xi : \xi \in \lambda\}$ (λ : a set) of commutative semigroups S_ξ , where each S_ξ satisfies $P(G)$, every semilattice composition of $\{S_\xi : \xi \in \lambda\}$ is commutative.

If (1. 9) or (1. 10) is true for $P(G)$, then $P(G)$ is called a linearly c-invariant property (abbrev., l.c.i.-property) or a fully c-invariant property (abbrev., f.c.i.-property) respectively. It is obvious from the definitions that an l.c.e. [f.c.e.] -property is an l.c.i. [f.c.i.] -property. In § 4, we shall show existence of maximal l.c.i.-properties and maximal f.c.i.-properties and determine some of them.

Remark. Let $P_u(G)$ be an abstract property as follows :

(1. 11) G is universal, i.e., $G^2 = G$.

Then it is easy to see that $P_u(G)$ is an l.c.e.-property (this will be shown later). Now, let T be a universal commutative semigroup which has a zero element and whose annihilator A contains a non-zero element. (Existence of such a semigroup T can be proved). Now, let $L_3 = \{\alpha, \beta, \gamma\}$ be a semilattice such that $\alpha < \gamma$, $\beta < \gamma$, $\alpha \not\leq \beta$ and $\beta \not\leq \alpha$. Let S_α and S_β be infinite cyclic semigroups generated by a and b respectively : $S_\alpha = (a)$ and $S_\beta = (b)$. Let $S_\gamma = T$. Then $S = S_\alpha + S_\beta + S_\gamma$ becomes a non-commutative composition of $\{S_\alpha, S_\beta, S_\gamma\}$ with respect to L_3 by multiplication \circ defined as follows :

$$x \circ y = \begin{cases} xy & \text{if } x, y \in S_\alpha, \in S_\beta \text{ or } \in S_\gamma, \\ u & \text{if } x=a \text{ and } y=b, \\ 0 & \text{otherwise,} \end{cases}$$

where 0 is the zero element of $T (=S_\gamma)$ and u is a fixed non-zero element contained in A . Hence $P_u(G)$ is not an f.c.e.-property.

Notation. Throughout this paper, if $\{S_\xi : \xi \in X\}$ is a collection of commutative semigroups S_ξ , we shall denote elements of S_ξ by small letters a_ξ, b_ξ, c_ξ etc. having ξ as their subscripts.

§ 2. Composition theorems. Let $\mathcal{A} = \{S_\tau : \tau \in \Gamma\}$ (Γ : a semi-lattice) be a collection of commutative semigroups S_τ . For every pair (α, β) of $\alpha, \beta \in \Gamma$ with $\alpha \leq \beta$, let $\mathcal{M}(\alpha, \beta)$ be the set of mappings of S_α into S_β . Let $C(\alpha, \beta) = \{S_\xi : \xi \in \Gamma, \alpha \leq \xi \leq \beta\}$. Clearly $S_\beta \in C(\alpha, \beta)$. For every $S_\xi \in C(\alpha, \beta)$, let ψ_ξ, φ_ξ be (not necessarily distinct) two mappings of S_ξ into $\mathcal{M}(\alpha, \beta)$. Put $\psi_\xi(a_\xi) = \bar{a}_\xi^{(\alpha, \beta)}$ and $\varphi_\xi(a_\xi) = \tilde{a}_\xi^{(\alpha, \beta)}$. Let $\mathcal{M}_L(\mathcal{A}) \equiv \mathcal{M}_L(S_\tau : \tau \in \Gamma) = \{\bar{a}_\xi^{(\alpha, \beta)} : \alpha \leq \beta, \alpha, \beta \in \Gamma, a_\xi \in S_\xi, S_\xi \in C(\alpha, \beta)\}$, and $\mathcal{M}_R(\mathcal{A}) \equiv \mathcal{M}_R(S_\tau : \tau \in \Gamma) = \{\tilde{a}_\xi^{(\alpha, \beta)} : \alpha \leq \beta, \alpha, \beta \in \Gamma, a_\xi \in S_\xi, S_\xi \in C(\alpha, \beta)\}$.

If

$$(2.1) \quad \mathcal{M}(\mathcal{A}) \equiv \mathcal{M}(S_\tau : \tau \in \Gamma) = \mathcal{M}_L(\mathcal{A}) + \mathcal{M}_R(\mathcal{A})$$

satisfies the following condition (C), then $\mathcal{M}(\mathcal{A})$ is called a set of composite factors on \mathcal{A} :

$$(C) \quad \left\{ \begin{array}{l} (1) \quad \bar{a}_\alpha^{(\beta, \alpha\beta)} \sim_{C_\tau}^{(\alpha\beta, \alpha\beta\tau)} \sim_{C_\tau}^{(\beta, \beta\tau)} \bar{a}_\alpha^{(\beta\tau, \alpha\beta\tau)}, \\ (2) \quad \bar{a}_\alpha^{(\alpha, \alpha)} = \tilde{a}_\alpha^{(\alpha, \alpha)} = \text{the inner translation } \rho_{a_\alpha} \text{ on } S_\alpha \text{ induced by } a_\alpha, \\ (3) \quad \bar{a}_\alpha^{(\beta, \alpha\beta)} \text{ and } \tilde{b}_\beta^{(\alpha, \alpha\beta)} \text{ are conjugate to each other in the following sense : } \bar{a}_\alpha^{(\beta, \alpha\beta)}(b_\beta) = \tilde{b}_\beta^{(\alpha, \alpha\beta)}(a_\alpha). \end{array} \right.$$

Theorem 1. Let $\mathcal{A} = \{S_\tau : \tau \in \Gamma\}$ (Γ : a semilattice) be a collection of commutative semigroups S_τ .

(i) \mathcal{A} is composable if and only if there exists a set of composite factors on \mathcal{A} .

(ii) Let $\mathcal{M}(\mathcal{A})$ of (2. 1) be a set of composite factors on \mathcal{A} . Then $S = \sum \{S_\gamma : \gamma \in \Gamma\}$ becomes a composition $S(\circ)$ of \mathcal{A} by multiplication \circ defined by

$$(2. 2) \quad a_\alpha \circ b_\beta = \overline{a}_\alpha^{(\beta, \alpha\beta)}(b_\beta) (= \widetilde{b}_\beta^{(\alpha, \alpha\beta)}(a_\alpha)).$$

Further, every possible composition of \mathcal{A} is found in this fashion.

The composition $S(\circ)$ in (ii) of Theorem 1 is called the composition of \mathcal{A} induced by $\mathcal{M}(\mathcal{A})$.

Corollary 1. Let $\mathcal{A} = \{S_\gamma : \gamma \in \Gamma\}$ (Γ : a semilattice) be a collection of commutative semigroups S_γ , and $\mathcal{M}(\mathcal{A})$ of (2. 1) a set of composite factors on \mathcal{A} . Then, the composition $S(\circ)$ of \mathcal{A} induced by $\mathcal{M}(\mathcal{A})$ is non-commutative if and only if the following condition is satisfied :

$$(2. 3) \quad \overline{a}_\alpha^{(\beta, \alpha\beta)} \neq \widetilde{a}_\alpha^{(\beta, \alpha\beta)} \text{ for some } a_\alpha \in S_\alpha, \alpha, \beta \in \Gamma.$$

Corollary 2. Let $\mathcal{A} = \{S_\gamma : \gamma \in \Gamma\}$ (Γ : a semilattice) be a collection of commutative semigroups S_γ . Then, every composition of \mathcal{A} is commutative if and only if there is no set, $\mathcal{M}(\mathcal{A})$ of (2. 1), of composite factors on \mathcal{A} which satisfies the condition (2. 3).

Now, as a special case, we consider a collection $\mathcal{A} = \{S_\gamma : \gamma \in \Gamma\}$ of commutative semigroups S_γ having a chain Γ as its index set. Let $\mathcal{M}(\mathcal{A})$ of (2. 1) be a set of composite factors on \mathcal{A} .

Then, we have the following lemmas :

Lemma 1. For $\alpha, \beta \in \Gamma$ with $\alpha \leq \beta$, each of $\overline{a}_\alpha^{(\beta, \alpha\beta)} (= \overline{a}_\alpha^{(\beta, \beta)})$ and $\widetilde{a}_\alpha^{(\beta, \alpha\beta)} (= \widetilde{a}_\alpha^{(\beta, \beta)})$ is a translation on S_β .

Lemma 2. For $\alpha, \beta \in \Gamma$ with $\alpha \geq \beta$, $\overline{a}_\alpha^{(\beta, \alpha)}(b_\beta) = \widetilde{b}_\beta^{(\alpha, \alpha)}(a_\alpha)$ and $\widetilde{b}_\alpha^{(\beta, \alpha)}(a_\beta) = \overline{a}_\beta^{(\alpha, \alpha)}(b_\alpha)$.

Putting $\overline{a}_\alpha^{(\beta, \beta)} = \rho_{\alpha, \beta}$ and $\widetilde{a}_\alpha^{(\beta, \beta)} = \delta_{\alpha, \beta}$ for $\alpha \leq \beta$, we have

the following lemmas :

Lemma 3. $\rho_{a\alpha, \alpha} = \tilde{\sigma}_{a\alpha, \alpha}$ = the inner translation $\rho_{a\alpha}$ on S_α induced by $a\alpha$.

Lemma 4. $\rho_{a\alpha, \gamma} \tilde{\sigma}_{b\beta, \gamma} = \tilde{\sigma}_{b\beta, \gamma} \rho_{a\alpha, \gamma}$ if $\alpha \leq \gamma, \beta \leq \gamma$.

Lemma 5.
$$\rho_{b\beta, \gamma} \rho_{a\alpha, \gamma} = \begin{cases} \rho_{a\alpha, \beta(b\beta), \gamma} & \text{if } \alpha \leq \beta \leq \gamma, \\ \rho_{\tilde{\sigma}_{b\beta, \alpha}(a\alpha), \gamma} & \text{if } \beta \leq \alpha \leq \gamma. \end{cases}$$

Lemma 6.
$$\tilde{\sigma}_{b\beta, \gamma} \tilde{\sigma}_{a\alpha, \gamma} = \begin{cases} \tilde{\sigma}_{a\alpha, \beta(b\beta), \gamma} & \text{if } \alpha \leq \beta \leq \gamma, \\ \tilde{\sigma}_{\rho_{b\beta, \alpha}(a\alpha), \gamma} & \text{if } \beta \leq \alpha \leq \gamma. \end{cases}$$

By Lemmas 3 - 6, we obtain the following result : Let $\Omega = \{S_\gamma : \gamma \in \Gamma\}$ (Γ : a chain) be a collection of commutative semigroups S_γ , $\mathcal{M}(\Omega)$ of (2.1) a set of composite factors on Ω . Let $S(\circ)$ be the composition of $\{S_\gamma : \gamma \in \Gamma\}$ induced by $\mathcal{M}(\Omega)$.

Then, there exists a system

(2.4) $\mathcal{G}(\Omega) = \{\rho_{a\alpha, \beta} : a\alpha \in S_\alpha, \alpha \leq \beta, \alpha, \beta \in \Gamma\} \dot{+} \{\tilde{\sigma}_{a\alpha, \beta} : a\alpha \in S_\alpha, \alpha \leq \beta, \alpha, \beta \in \Gamma\}$, where $\rho_{a\alpha, \beta}$ and $\tilde{\sigma}_{a\alpha, \beta}$ are mappings of S_β into S_β , such that

$$(T) \quad \begin{cases} (1) \rho_{a\alpha, \beta} \text{ and } \tilde{\sigma}_{a\alpha, \beta} \text{ are translations on } S_\beta, \\ (2) \rho_{a\alpha, \alpha} = \tilde{\sigma}_{a\alpha, \alpha} = \text{the inner translation } \rho_{a\alpha} \text{ induced by } a\alpha \\ (3) \rho_{a\alpha, \gamma} \tilde{\sigma}_{b\beta, \gamma} = \tilde{\sigma}_{b\beta, \gamma} \rho_{a\alpha, \gamma}, \\ (4) \rho_{b\beta, \gamma} \rho_{a\alpha, \gamma} = \begin{cases} \rho_{a\alpha, \beta(b\beta), \gamma} & \text{if } \alpha \leq \beta \leq \gamma, \\ \rho_{\tilde{\sigma}_{b\beta, \alpha}(a\alpha), \gamma} & \text{if } \beta \leq \alpha \leq \gamma, \end{cases} \\ (5) \tilde{\sigma}_{b\beta, \gamma} \tilde{\sigma}_{a\alpha, \gamma} = \begin{cases} \tilde{\sigma}_{a\alpha, \beta(b\beta), \gamma} & \text{if } \alpha \leq \beta \leq \gamma, \\ \tilde{\sigma}_{\rho_{b\beta, \alpha}(a\alpha), \gamma} & \text{if } \beta \leq \alpha \leq \gamma. \end{cases} \end{cases}$$

Further, the multiplication \circ in $S(\circ)$ is represented by

$$(P) \quad a\alpha \circ b\beta = \begin{cases} \bar{a}_\alpha^{(\beta, \beta)}(b\beta) = \rho_{a\alpha, \beta}(b\beta) & \text{if } \alpha \leq \beta, \\ \tilde{b}_\beta^{(\alpha, \alpha)}(a\alpha) = \tilde{\sigma}_{b\beta, \alpha}(a\alpha) & \text{if } \alpha \geq \beta. \end{cases}$$

In general, let $\mathcal{A} = \{S_\gamma : \gamma \in \Gamma\}$ (Γ : a chain) be a collection of commutative semigroups S_γ . For each pair (α, β) , where $\alpha \in S_\alpha$, $\alpha \leq \beta$ and $\alpha, \beta \in \Gamma$, let $\rho_{\alpha, \beta}$ and $\tilde{\rho}_{\alpha, \beta}$ be (not necessarily distinct) two mappings of S_β into S_α . If $\mathcal{E}(\mathcal{A}) = \{\rho_{\alpha, \beta} : \alpha \in S_\alpha, \alpha \leq \beta, \alpha, \beta \in \Gamma\} \cup \{\tilde{\rho}_{\alpha, \beta} : \alpha \in S_\alpha, \alpha \leq \beta, \alpha, \beta \in \Gamma\}$ satisfies (T) of (2. 4), then $\mathcal{E}(\mathcal{A})$ is called a factor set of translations on \mathcal{A} . From this

definition and the above-mentioned result, we can conclude as follows :

Let $\mathcal{A} = \{S_\gamma : \gamma \in \Gamma\}$ (Γ : a chain) be a collection of commutative semigroups S_γ . If $S(\circ) = \sum \{S_\gamma : \gamma \in \Gamma\}$ is a composition of $\mathcal{A} = \{S_\gamma : \gamma \in \Gamma\}$, then there exists a factor set of translations on \mathcal{A} , say $\mathcal{E}(\mathcal{A}) = \{\rho_{\alpha, \beta} : \alpha \in S_\alpha, \alpha \leq \beta, \alpha, \beta \in \Gamma\} \cup \{\tilde{\rho}_{\alpha, \beta} : \alpha \in S_\alpha, \alpha \leq \beta, \alpha, \beta \in \Gamma\}$, and \circ in $S(\circ)$ is represented by (P).

Conversely, let $\mathcal{E}(\mathcal{A}) = \{\rho_{\alpha, \beta} : \alpha \in S_\alpha, \alpha \leq \beta, \alpha, \beta \in \Gamma\} \cup \{\tilde{\rho}_{\alpha, \beta} : \alpha \in S_\alpha, \alpha \leq \beta, \alpha, \beta \in \Gamma\}$ (Γ : a chain) be a factor set of translations on a collection $\mathcal{A} = \{S_\gamma : \gamma \in \Gamma\}$ of commutative semigroups S_γ . Then, we can prove that $S = \sum \{S_\gamma : \gamma \in \Gamma\}$ becomes a composition of \mathcal{A} by multiplication \circ given by (P).

Summarizing the results above, we obtain the following

Theorem 2. Let $\mathcal{A} = \{S_\gamma : \gamma \in \Gamma\}$ (Γ : a chain) be a collection of commutative semigroups S_γ . Let $\mathcal{E}(\mathcal{A})$ of (2. 4) be a factor set of translations on \mathcal{A} . Then $S = \sum \{S_\gamma : \gamma \in \Gamma\}$ becomes a composition $S(\circ)$ of \mathcal{A} by the multiplication \circ defined by (P). Further, every composition of \mathcal{A} is found in this fashion.

This result is a special case of Theorem 2. 1 given by Yoshida [5]. $S(\circ)$ in Theorem 2 is called the composition of \mathcal{A} induced by $\mathcal{E}(\mathcal{A})$.

From Theorem 2, we obtain immediately the following

Corollary 1. Let $\mathcal{A} = \{S_\gamma : \gamma \in \Gamma\}$ (Γ : a chain) be a collection of commutative semigroups S_γ , and $\mathcal{E}(\mathcal{A})$ of (2. 4) a factor set of

translations on \mathcal{A} . Then, the composition $S(\circ)$ of \mathcal{A} induced by $\mathcal{E}(\mathcal{A})$ is non-commutative if and only if

$$(2.5) \quad \rho_{\alpha,\beta} \neq \delta_{\alpha,\beta} \text{ for some } a_\alpha \in S_\alpha, \alpha, \beta \in \Gamma, \alpha < \beta.$$

Moreover, the following is obvious from Corollary 1 :

Corollary 2. Let $\mathcal{A} = \{S_\gamma : \gamma \in \Gamma\}$ (Γ : a chain) be a collection of commutative semigroups S_γ . Every composition of \mathcal{A} is commutative if and only if there is no factor set, $\mathcal{E}(\mathcal{A})$ of (2.4), of translations on \mathcal{A} which satisfies (2.5).

In the case where every composition of a collection $\mathcal{A} = \{S_\gamma : \gamma \in \Gamma\}$ (Γ : a chain) of commutative semigroups S_γ is commutative, we have another construction theorem for the compositions of \mathcal{A} which is somewhat simpler than Theorem 2 :

Theorem 3. Let $\mathcal{A} = \{S_\gamma : \gamma \in \Gamma\}$ (Γ : a chain) be a collection of commutative semigroups S_γ every composition of which is commutative. Let $S = \sum \{S_\gamma : \gamma \in \Gamma\}$. For each pair (a_α, β) , where $a_\alpha \in S_\alpha$, $\alpha, \beta \in \Gamma$ and $\alpha \leq \beta$, let $\rho_{a_\alpha, \beta}$ be a mapping of S_β into S_β . Let $\overline{\mathcal{E}}(\mathcal{A}) = \{\rho_{a_\alpha, \beta} : a_\alpha \in S_\alpha, \alpha \leq \beta, \alpha, \beta \in \Gamma\}$.

If $\overline{\mathcal{E}}(\mathcal{A})$ satisfies the condition

$$(\overline{T}) \quad \begin{cases} (1) \quad \rho_{a_\alpha, \beta} \text{ is a translation on } S_\beta, \\ (2) \quad \rho_{a_\alpha, \alpha} = \text{the inner translation } \rho_{a_\alpha} \text{ on } S_\alpha \text{ induced by } a_\alpha, \\ (3) \quad \rho_{a_\alpha, \gamma} \rho_{b_\beta, \gamma} = \rho_{b_\beta, \gamma} \rho_{a_\alpha, \gamma} = \rho_{\rho_{a_\alpha, \beta}(b_\beta), \gamma} \quad \text{if } \alpha \leq \beta \leq \gamma, \end{cases}$$

then S becomes a composition $S(\circ)$ of \mathcal{A} by the multiplication \circ defined by

$$(\overline{P}) \quad a_\alpha \circ b_\beta = b_\beta \circ a_\alpha = \rho_{a_\alpha, \beta}(b_\beta) \quad \text{if } \alpha \leq \beta.$$

Further, every composition of \mathcal{A} is found in this fashion.

Next, we present some results concerning a factor set of translations on a collection $\mathcal{A} = \{S_\gamma : \gamma \in \Gamma\}$ (Γ : a chain) of commutative semigroups S_γ .

Lemma 7. Let $\mathcal{A} = \{ S_\gamma : \gamma \in \Gamma \}$ (Γ : a chain) be a collection of commutative semigroups S_γ . Let $\mathcal{E}(\mathcal{A})$ of (2.4) be a factor set of translations on \mathcal{A} which satisfies (T) in (2.4). Then,

- (i) for any $a_\alpha \in S_\alpha$, $\alpha, \beta \in \Gamma$ with $\alpha \leq \beta$, $y_\beta \rho_{\alpha, \beta}(x_\beta) = \tilde{\sigma}_{\alpha, \beta}(y_\beta)x_\beta$, i.e., $\rho_{\alpha, \beta}$ and $\tilde{\sigma}_{\alpha, \beta}$ are linked, and
- (ii) $\rho_{\alpha, \beta} | s_\beta^2 = \tilde{\sigma}_{\alpha, \beta} | s_\beta^2$.

A commutative semigroup S is said to be reductive if it satisfies the following abstract property $P_r(G)$:

Reductivity $P_r(G)$: $ax = bx$ for all $x \in G$ implies $a = b$.

Lemma 8. Let \mathcal{A} and $\mathcal{E}(\mathcal{A})$ be as in Lemma 7. If each of the multiple-components of \mathcal{A} is universal or reductive, then $\rho_{\alpha, \beta} = \tilde{\sigma}_{\alpha, \beta}$ for every $a_\alpha \in S_\alpha$, $\alpha, \beta \in \Gamma$ with $\alpha \leq \beta$.

By using Lemma 8 and Corollary 2 to Theorem 2, we obtain

Corollary. Let $\mathcal{A} = \{ S_\gamma : \gamma \in \Gamma \}$ (Γ : a chain) be a collection of commutative semigroups S_γ . If each of the multiple-components of \mathcal{A} is universal or reductive, then every composition of \mathcal{A} is commutative.

Remark. This result will be more generalized in the next section.

§ 3. The weakest f.c.e. [l.c.e.] -property. In this section, we investigate l.c.e.-properties and f.c.e.-properties.

Let us consider the following abstract property $P_q(G)$ pertaining to commutative semigroups G :

- (3.1) There is no system $\{\tilde{\sigma}, \rho\}$ of distinct two translations on G such that (1) $\tilde{\sigma}\rho = \rho\tilde{\sigma}$ and (2) $\tilde{\sigma} | g^2 = \rho | g^2$.

This property $P_q(G)$ is called quasi-reductivity. As is shown later, reductivity implies quasi-reductivity. However, the converse is not true.

Lemma 9. Let $\{\tilde{\rho}, \rho\}$ be a system of distinct two translations $\tilde{\rho}, \rho$ on a commutative semigroup S such that $\tilde{\rho}\rho = \rho\tilde{\rho}$ and $\tilde{\rho}|S^2 = \rho|S^2$. Then there exist distinct two elements $x, y \in S$ and a prime element $t \in S$ such that (1) $xa = ya$ for all $a \in S$ and (2) $\tilde{\rho}(t) = x$ and $\rho(t) = y$.

(Note : An element of $S \setminus S^2$ is called prime)

It is easily seen from Lemma 9 that reductivity implies quasi-reductivity.

Example. Let $S = \{a, a^2, \dots, a^n\}$ be a cyclic semigroup of order n such that $n \geq 2$, $a^{n-1} \neq a^n$ and $aa^n = a^n$. Define mappings $\rho, \tilde{\rho}: S \rightarrow S$ as follows : $\rho(a) = a^{n-1}$, $\rho(a^i) = a^n$ if $i > 1$; and $\tilde{\rho}(a^i) = a^n$ for all i . Then $\rho, \tilde{\rho}$ are translations on S such that $\tilde{\rho}\rho = \rho\tilde{\rho}$ and $\tilde{\rho}|S^2 = \rho|S^2$. Hence, of course, S is not quasi-reductive.

By using Lemma 9, we can prove the following theorem :

Theorem 4. $P_q(G)$ is the weakest l.c.e.-property.

In general, it is easy to see that if $P(G)$ and $P_1(G)$ are abstract properties such that $P_1(G) \leq P(G)$ and if $P(G)$ is an l.c.e.-property, then $P_1(G)$ is also an l.c.e.-property. For abstract properties $P_1(G)$ and $P_2(G)$, denote the property " $P_1(G)$ or $P_2(G)$ " by $P_1(G) \vee P_2(G)$. It is obvious that $P_1(G) \leq P_1(G) \vee P_2(G)$ and $P_2(G) \leq P_1(G) \vee P_2(G)$. Now, it is easy to see that $P_r(G) \vee P_u(G) \leq P_q(G)$. Since $P_r(G) \leq P_r(G) \vee P_u(G)$ and $P_u(G) \leq P_r(G) \vee P_u(G)$, we have $P_r(G) \leq P_q(G)$ and $P_u(G) \leq P_q(G)$. Since $P_q(G)$ is an l.c.e.-property, each of $P_r(G)$, $P_u(G)$ and $P_r(G) \vee P_u(G)$ is also an l.c.e.-property.

Thus, we have the following result as a corollary to Theorem 4 :

Corollary. Each of reductivity, universality and the property "reductive or universal" is an l.c.e.-property.

Remarks. (1) Moreover, the following is obvious from Theorem 4 :

Let $\mathcal{L} = \{S_\xi : \xi \in \mathcal{X}\}$ (\mathcal{X} : a set) be a collection of commutative semigroups S_ξ , where $P_q(S_\xi)$ is true for all $S_\xi \in \mathcal{L}$. Then, every linear composition of \mathcal{L} is commutative.

(2) For a special collection $\mathcal{L} = \{S_\tau : \tau \in \Gamma\}$ (Γ ; a chain) of commutative semigroups S_τ , every composition of \mathcal{L} is commutative even if there exists a multiple-component S_τ which does not satisfy $P_q(G)$. For example, let $L = \{0, 1\}$ be a chain with respect to the usual multiplication, $S_1 = \{e\}$ a semigroup consisting of a single element e and $S_0 = \{a, a^2, \dots, a^{n-1}, a^n\}$ a cyclic semigroup of order n ($n > 2$) such that $a^{n-1} \neq a^n$ and $aa^n = a^n$. Then, it is easy to see from the above-mentioned example that $P_q(S_0)$ is not true. However, there is no non-commutative composition of $\{S_1, S_0\}$ with respect to L .

Hereafter, for any element x of a commutative semigroup S , the inner translation on S induced by x will be denoted by ρ_x .

Now, let us consider the following abstract property $P_T^*(G)$ pertaining to commutative semigroups G :

(3. 2) There is no system $\{u, v ; \xi, \eta\}$ of distinct two elements u, v of G and (not necessarily distinct) translations ξ, η on G such that (1) $\xi\eta = \eta\xi = \rho_u = \rho_v$ and (2) $\xi(u) = \xi(v)$ and $\eta(u) = \eta(v)$.

At first, we have

Lemma 10. Let $\{u, v ; \xi, \eta\}$ be a system of distinct two elements u, v of a commutative semigroup S and translations ξ, η on S , satisfying (1), (2) of (3. 2). Then, $uz = vz$ for all $z \in S$.

The following lemma was shown by R. Yoshida, though the result is not yet published :

Lemma 11. $P_T^*(G)$ is equivalent to $P_T(G)$.

By using Lemmas 10 and 11, we can prove the following theorem which is one of the main results of this paper :

Theorem 5. $P_r(G)$ is the weakest f.c.e.-property.

Corollary. Let $\mathcal{A} = \{S_\xi : \xi \in \mathcal{X}\}$ (\mathcal{X} : a set) be a collection of commutative semigroups S_ξ , where $P_r(S_\xi)$ is true for every $S_\xi \in \mathcal{A}$. Then, every semilattice composition of \mathcal{A} is commutative.

Remarks. (1) It is easy to see that if $P_0(G)$ is an f.c.e. [l.c.e.] -property and if $P(G)$ is an abstract property such that $P(G) \leq P_0(G)$, then $P(G)$ is also an f.c.e. [l.c.e.] -property. Let $\mathcal{P}(Q) = \{P(G) : P(G) \text{ is an abstract property such that } P(G) \leq P_Q(G)\}$ and $\mathcal{P}(R) = \{P(G) : P(G) \text{ is an abstract property such that } P(G) \leq P_r(G)\}$. Then, $\mathcal{P}(Q)$ and $\mathcal{P}(R)$ are the set of all l.c.e.-properties and the set of all f.c.e.-properties respectively.

(2) As was shown in §1, there exists a universal commutative semigroup S which has a zero element 0 and whose annihilator A contains a non-zero element. Since $P_Q(S)$ is true and $P_r(S)$ is not true, $P_Q(G) \not\equiv P_r(G)$. Hence, $P_Q(G) > P_r(G)$. This also means that quasi-reductivity does not imply reductivity.

(3) Since $P_r(G)$ is weaker than each of separativity (see Clifford & Preston [2]) and cancellativity, the following results immediately follow from the above-mentioned Corollary :

(i) A semigroup which is a semilattice of commutative reductive semigroups is commutative and reductive.

(ii) A semigroup which is a semilattice of separative commutative semigroups is separative and commutative.

(iii) A semigroup which is a semilattice of cancellative commutative semigroups is separative and commutative.

The converse of this result also holds (see Clifford & Preston [2]) ; i.e., a separative commutative semigroup is a semilattice of cancellative commutative semigroups.

§5. Maximal f.c.i. [l.c.i.] -properties. Let $\mathcal{F} = \{P_\lambda(G) : \lambda \in \Lambda\}$ be the set of all f.c.i.-properties $P_\lambda(G)$. Then \mathcal{F} is clearly a partially ordered set with respect to the ordering relation \leq defined by (1.6). (Recall that equivalent two properties are regarded as the same property). Let $\mathcal{T} = \{P_\tau(G) : \tau \in \Lambda_0\}$ be any totally ordered subset of \mathcal{F} . Define an abstract property $T(G)$ as follows : $T(G) = \bigvee_{\tau \in \Lambda_0} P_\tau(G)$, i.e., $T(G)$ = the property "being at least one of $\{P_\tau(G) : \tau \in \Lambda_0\}$ ". Hence, a commutative semigroup S satisfies $T(G)$ if and only if S satisfies at least one of the properties $\{P_\tau(G) : \tau \in \Lambda_0\}$. Now, let $\mathcal{L} = \{S_\xi : \xi \in \mathcal{X}\}$ (\mathcal{X} : a set) be a collection of commutative semigroups S_ξ such that every S_ξ satisfies $T(G)$. Suppose that there exists a non-commutative semilattice composition $S(\circ) = \sum \{S_\xi : \xi \in \mathcal{X}(\ast)\}$ of \mathcal{L} . Then there exist a, b such that $a \in S_\tau, b \in S_\delta, \tau, \delta \in \mathcal{X}$ and $a \circ b \neq b \circ a$. Clearly, both $a \circ b$ and $b \circ a$ are contained in $S_{\tau \ast \delta}$. Put $S_\tau + S_\delta + S_{\tau \ast \delta} = M$. Then $M(\circ)$ is a subsemigroup of $S(\circ)$ and is non-commutative. Since $T(S_\tau), T(S_\delta)$ and $T(S_{\tau \ast \delta})$ are all true, there exist $P_\alpha(G), P_\beta(G)$ and $P_\epsilon(G)$ of $\{P_\tau(G) : \tau \in \Lambda_0\}$ such that $P_\alpha(S_\tau), P_\beta(S_\delta)$ and $P_\epsilon(S_{\tau \ast \delta})$ are true. Let $P_\gamma(G)$ be the weakest property in $\{P_\alpha(G), P_\beta(G), P_\epsilon(G)\}$. Then $P_\gamma(G)$ is of course an f.c.i.-property and $P_\gamma(S_\tau), P_\gamma(S_\delta), P_\gamma(S_{\tau \ast \delta})$ are all true. Hence, the semilattice composition $M(\circ)$ of $\{S_\tau, S_\delta, S_{\tau \ast \delta}\}$ must be commutative. However, this is a contradiction since $M(\circ)$ was non-commutative. Consequently, every semilattice composition of \mathcal{L} must be commutative. Therefore, $T(G)$ is an f.c.i.-property and hence $T(G) \in \mathcal{F}$. Since $P_\tau(G) \leq T(G)$ for all $\tau \in \Lambda_0$, $T(G)$ is an upper bound of \mathcal{T} . Thus, \mathcal{F} is an inductively ordered set. Hence, there exists a maximal f.c.i.-property in \mathcal{F} . Existence of maximal l.c.i.-properties is also proved by a similar method.

Hence, we have

Theorem 6. There exist a maximal l.c.i.-property and a maximal f.c.i.-property.

Corollary. For any f.c.i. [l.c.i.] -property $P(G)$, there exists a maximal f.c.i. [l.c.i.] -property $P_m(G)$ such that $P(G) \leq P_m(G)$.

In fact, the following three theorems show that quasi-reductivity is a maximal l.c.i.-property and both universality and reductivity are maximal f.c.i.-properties :

Theorem 7. $P_q(G)$ is a maximal l.c.i.-property.

Theorem 8. $P_r(G)$ is a maximal f.c.i.-property.

Theorem 9. $P_u(G)$ is a maximal f.c.i.-property.

From Theorem 9, we also have immediately

Corollary. A semigroup which is a semilattice of universal commutative semigroups is universal and commutative.

Remark. Let $\mathcal{F}[\mathcal{L}]$ be the set of all f.c.i. [l.c.i.] -properties. For $P_1(G), P_2(G) \in \mathcal{F}[\mathcal{L}]$, let us define an abstract property $P_1(G) \wedge P_2(G)$ as follows :

(4. 1) $P_1(G) \wedge P_2(G) =$ the property " being both $P_1(G)$ and $P_2(G)$ ".
Then, it is easy to see that $P_1(G) \wedge P_2(G) \in \mathcal{F}[\mathcal{L}]$ for any $P_1(G), P_2(G) \in \mathcal{F}[\mathcal{L}]$ and $P_1(G) \wedge P_2(G)$ is the greatest lower bound of $P_1(G)$ and $P_2(G)$. Further, in fact $\mathcal{F}[\mathcal{L}]$ is a semilattice with respect to this operation \wedge .

Since $P_u(G)$ and $P_r(G)$ are non-equivalent maximal f.c.i.-properties, it is obvious that there is no greatest f.c.i.-property, i.e., there is no weakest f.c.i.-property $P_g(G)$ in the following sense :

(4. 2) $P_g(G) \geq P(G)$ for any f.c.i.-property $P(G)$.

However, the author can not solve the following two problems and leaves them as open problems :-

Problem 1. Is there a maximal l.c.i.-property except $P_q(G)$?
 That is : Is $P_q(G)$ the greatest (weakest) l.c.i.-property ?
 Determine all of the maximal l.c.i.-properties.

Problem 2. Is there a maximal f.c.i.-property except $P_u(G)$ and $P_r(G)$? Determine all of the maximal f.c.i.-properties.

As a partial solution of Problem 1, we obtain the following result :
 Let C be an infinite cyclic semigroup : $C = \{a, a^2, \dots, a^n, \dots\}$.
 Let C^1 be the adjunction of an identity element to C : $C^1 = C \cup \{1\}$.
 Let $\mathcal{L}^* = \{P(G) : P(G) \text{ is an abstract property such that } P(G) \leq L(G),$
 for some l.c.i.-property $L(G)$ satisfied by $C^1\}$. Then $\mathcal{L}^* \ni P_q(G),$
 $P_r(G), P_u(G)$, since each of $P_q(C^1), P_u(C^1)$ and $P_r(C^1)$ is true and
 each of $P_q(G), P_u(G)$ and $P_r(G)$ is an l.c.i.-property. Further, \mathcal{L}^*
 $\supset \bar{\mathcal{L}} = \{P(G) : P(G) \text{ is an l.c.i.-property which is comparable with}$
 $P_r(G) \text{ or } P_u(G)\}$. In fact, let $P(G)$ be a property of $\bar{\mathcal{L}}$. If $P(G) \leq$
 $P_r(G)$ or $\leq P_u(G)$, then $P(G) \in \mathcal{L}^*$ since each of $P_r(G)$ and $P_u(G)$ is
 an l.c.i.-property and is satisfied by C^1 . If $P(G) > P_r(G)$ or $>$
 $P_u(G)$, then $P(C^1)$ is true since each of $P_r(C^1)$ and $P_u(C^1)$ is true.
 Since $P(C^1)$ is true and $P(G)$ is an l.c.i.-property, $P(G)$ is also
 contained in \mathcal{L}^* . In any cases, $P(G) \in \mathcal{L}^*$. Therefore, $\bar{\mathcal{L}} \subset \mathcal{L}^*$.
 Especially, cancellativity, separativity, regularity and the property
 " being a commutative semigroup G with 1 " are all contained in \mathcal{L}^* .

Now, we have

Theorem 10. $P_q(G)$ is the greatest (i.e., weakest) l.c.i.-property
 in \mathcal{L}^* .

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